quency vs gap-distance curve is not affected greatly by boring holes in the endplate and center conductor; thus, the compensation would not be affected markedly by these modifications.

Results

Temperature Compensation

A cavity was constructed with the following dimensions: $b=0.250$, $a=0.840$, $L_g=0.675$, and $L_p=7.025$ (all dimensions in inches). The cavity had a resonant frequency of 403.5 mc, 0.2 per cent higher than (13a) would indicate, and an unloaded Q of 2280.

The temperature compensation of the cavity was measured by placing it in a constant-temperature bath and measuring its resonant frequency as a function of temperature. The cavity temperature was varied from $-32°$C to 53°C, and the temperature coefficient of the cavity was found to be 2.62 parts/million/C° undercompensation; *i.e.*, the cavity frequency was lowered as the temperature was raised. This was a rather surprising result since an uncompensated invar cavity would be expected to exhibit a temperature coefficient of 1×10^{-6}/C° undercompensation. The high coefficient indicated one of two things: either 1) the invar center conductor had a higher expansion coefficient than advertised, or 2) some unexpected effect (such as eccentricity of the center conductor, for example) was present. Whatever the cause of the undercompensation, the gap length had to be shortened to compensate for it. This was done in steps until a final gap length of 0.376 inch produced a temperature coefficient of 0.25×10^{-6}/C° overcompensation, which was considered satisfactory. Fig. 9 shows the effects of the various gap lengths on the compensation of the cavity.

Transient Response

During the temperature tests some data on the transient response of the cavity were obtained. The cavity experienced a rise in temperature when it was inserted into the temperature bath. The response of the cavity showed two time constants, one associated with the transfer of heat to the outer conductor and the other associated with the transfer of heat to the center conductor. These time constants were measured to be 1.1 minute and 29 minutes, respectively. These results agree fairly well with the response determined analytically. As shown in Fig. 10, the cavity reached thermal equilibrium in about 2 hours after it experienced a change in temperature.

It should be mentioned that these figures for the transient response are not those which would be experienced in actual use. When employed in atmospheric soundings, the rate of transfer of heat from the air to the cavity would be very slow so that it would be expected that the cavity would have essentially the same temperature throughout the center post and the body at a given time. Hence, the temperature compensation techniques described in this paper would be effective.

A Graphical Method for Measuring Dielectric Constants at Microwave Frequencies*

CHARLES B. SHARPE†

Summary—**This paper describes a graphical method for measuring the real and imaginary parts of the dielectric constant $\epsilon/\epsilon_0=\epsilon'-j\epsilon''$ of materials at microwave frequencies. The method is based on the network approach to dielectric measurements proposed by Oliner and Altschuler in which the dielectric sample fills a section of transmission line or waveguide. In contrast to their method, the network representing the dielectric sample is analyzed in terms of the bilinear transformation**

$$\Gamma'=\frac{a\Gamma+b}{c\Gamma+d}; \qquad ad-bc=4.$$

The analysis proceeds from the geometric properties of the image circle in the Γ plane obtained by terminating the output line in a calibrated sliding short.

The technique described retains the desirable features of the network approach but avoids the necessity of measuring both scattering coefficients. As a result the procedure is more direct and, in the case of the TEM configuration, leads to an entirely graphical solution in which the complex dielectric constant can be read from a Smith chart overlay.

* Manuscript received by the PGMTT, August 7, 1959; revised manuscript received October 9, 1959. This work was sponsored by the U. S. Army Signal Res. and Dev. Lab., Fort Monmouth, N. J.
† The University of Michigan Res. Inst., Ann Arbor, Mich.

Introduction

There are many techniques for making dielectric measurements at microwave frequencies.[1] One of the more interesting methods proposed in recent years is that due to Oliner and Altschuler,[2] in which the

[1] A. von Hippel, ed., "Dielectric Materials and Applications," J. Wiley and Sons, Inc., New York, N. Y., ch. 2; 1954.

[2] A. Oliner and H. Altschuler, "Methods of measuring dielectric constants based upon a microwave network viewpoint," *J. Appl. Phys.*, vol. 26, pp. 214–219; February, 1955.

dielectric sample filling a section of waveguide is represented by a two-port microwave network as illustrated in Fig. 1. In their method the scattering matrix of the network is determined at reference planes T_1 and T_2 by Deschamps[3] procedure or, when the network can be regarded as lossless, by alternative so-called precision techniques. The complex relative dielectric constant $\epsilon/\epsilon_0 = \epsilon' - j\epsilon''$ is then obtained from either

$$\epsilon/\epsilon_0 = (Y/Y_0)^2 \text{ (TEM modes)} \tag{1a}$$

or

$$\epsilon/\epsilon_0 = \frac{(Y/Y_0)^2 + (\lambda_{0g}/\lambda_c)^2}{1 + (\lambda_{0g}/\lambda_c)^2} \text{ (H modes)}, \tag{1b}$$

where the wave admittance Y in the dielectric relative to the wave admittance of the empty guide Y_0 is given in terms of the scattering coefficients by

$$(Y/Y_0)^2 = \frac{(1 - S_{11})^2 - S_{12}^2}{(1 + S_{11})^2 - S_{12}^2}, \tag{2}$$

and λ_{0g} and λ_c refer to the guide wavelength and the cutoff wavelength, respectively, in the air-filled guide.

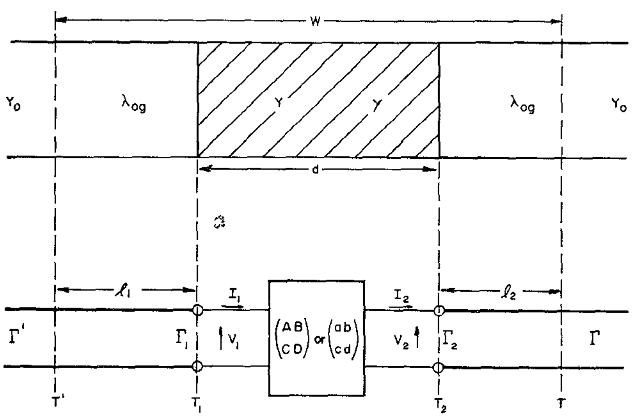

Fig. 1.—Dielectric sample in a waveguide and its equivalent circuit.

Oliner and Altschuler point out that the introduction of the network point of view to dielectric measurements results in two major advantages over earlier methods. First, it becomes possible to employ precision techniques in the determination of the network parameters. For example, in Deschamps' geometrical method, the image circle, representing the locus of points in the input reflection coefficient plane as a sliding short is moved in the output waveguide, is determined by graphical averaging. Therefore, the center of the image circle and its radius, as well as quantities derived from them, can be determined to a higher degree of precision than that of a single data point. The second feature of the network method which can be exploited to advantage in dielectric measurements concerns the concept of invariance. Briefly stated, invariance in the present case refers to the method of microwave network representation or measurement which calls for a minimum number of physical length measurements. Thus, for the configuration illustrated in Fig. 1 it is possible to take advantage of the known symmetry of the network to reduce to one the number of required distance measurements. The single measurement required may be either the length of the sample, d (location invariant) or the location of the one of the sample faces, T_1 or T_2 (length invariant). The desirability of employing a distance invariant method lies in the fact that errors arising from physical distance measurements are generally greater than those resulting from the electrical distance measurements, assuming that corrections have been made for errors in the location of the voltage minimum caused by spurious discontinuities if they exist.

[3] G. A. Deschamps, "Determination of reflection coefficients and insertion loss of a waveguide junction," *J. Appl. Phys.*, vol. 24, pp. 1046–1050; August, 1953.

The purpose of the present paper is to describe a technique for measuring dielectric constants which retains the desirable features of the network approach but which can be accomplished more directly and with a minimum of computation. In the case of the TEM configuration, the dielectric constant can be obtained by a purely graphical procedure in which the desired complex constant is read directly from a Smith Chart.

Theory

The dielectric-filled section of line or waveguide can be represented at reference planes T_1 and T_2 by the $(ABCD)$ circuit parameters which relate the input and output voltages and currents, as defined in Fig. 1, by the matrix equation

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix}, \tag{3}$$

where $AD - BC = 1$. If we take

$$Z_1 = V_1/I_1, \qquad Z_2 = V_2/I_2, \tag{4}$$

then a bilinear relationship is obtained between the impedances Z_1 and Z_2:

$$Z_1 = \frac{AZ_2 + B}{CZ_2 + D}. \tag{5}$$

This transformation has often been used in analyzing the properties of linear two-port networks. However, it will be more convenient in the present case to consider the network representing the dielectric sample in terms of a bilinear transformation in the reflection coefficient or Γ plane. Thus, if the input and output reflection coefficients are defined, respectively, by

$$\Gamma_1 = \frac{Y_0 - Y_1}{Y_0 + Y_1}, \qquad \Gamma_2 = \frac{Y_0 - Y_2}{Y_0 + Y_2}, \tag{6}$$

it can be shown that

$$\Gamma_1 = \frac{a\Gamma_2 + b}{c\Gamma_2 + d}, \tag{7}$$

where

$$a = A - BY_0 - C/Y_0 + D,$$
$$b = A + BY_0 - C/Y_0 - D,$$
$$c = A - BY_0 + C/Y_0 - D,$$
$$d = A + BY_0 + C/Y_0 + D. \quad (8)$$

Reciprocity is assured if $ad-bc=4$. If the network is symmetrical, as in the present case, then $b=-c$. Eq. (7) is sometimes called the direct transformation to distinguish it from the inverse transformation

$$\Gamma_2 = \frac{-d\Gamma_1 + b}{c\Gamma_1 - a}. \quad (9)$$

The matrix composed of the $(abcd)$ parameters might be termed the reflection matrix of the network. Although the transformation (7) has been used by Mathis[4] and also by Bolinder,[5] employing a different normalization, it has not enjoyed widespread use as a tool in network analysis. As would be expected, the reflection matrix bears a close connection to the scattering matrix. It can be shown that, in general,

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} = \begin{bmatrix} b/d & 2/d \\ (ad-bc)/2d & -c/d \end{bmatrix}. \quad (10)$$

An obvious application of (10) is suggested in the Appendix. The components of the reflection matrix transform in a manner very similar to the manner in which the scattering coefficients transform as a result of a shift in reference planes. Referring to Fig. 1, the reflection matrix at reference planes T' and T is given by

$$\begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} = \begin{bmatrix} e^{-j\phi_1} & 0 \\ 0 & e^{j\phi_1} \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e^{-j\phi_2} & 0 \\ 0 & e^{j\phi_2} \end{bmatrix}, \quad (11)$$

where $\phi_1 = 2\pi l_1/\lambda_{0g}$, $\phi_2 = 2\pi l_2/\lambda_{0g}$, and the primed coefficients are defined by

$$\Gamma' = \frac{a'\Gamma + b'}{c'\Gamma + d'}. \quad (12)$$

Returning to the problem at hand the $(ABCD)$ matrix of the dielectric-filled section at reference planes T_1 and T_2 is

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \cosh \gamma d & \dfrac{\sinh \gamma d}{Y} \\ Y \sinh \gamma d & \cosh \gamma d \end{bmatrix} \quad (13)$$

from which (8) yields,

$$a = 2\cosh \gamma d - (Y/Y_0 + Y_0/Y) \sinh \gamma d,$$
$$b = -c = -(Y/Y_0 - Y_0/Y) \sinh \gamma d,$$
$$d = 2\cosh \gamma d + (Y/Y_0 + Y_0/Y) \sinh \gamma d, \quad (14)$$

where $\gamma = \alpha + j\beta$ is the propagation constant in the dielectric. A purely algebraic relation between Y/Y_0 and the $(abcd)$ parameters can be obtained by forming

$$\Gamma_E \equiv \frac{a-d}{2c} = \frac{1 + (Y/Y_0)^2}{1 - (Y/Y_0)^2}. \quad (15)$$

This expression is entirely equivalent to (2) involving the scattering coefficients. The merit of the reflection parameter representation lies in the simplicity of (15), as well as in the facility it provides in the geometric interpretation of the problem. Thus, it will be shown that Γ_E can be determined through a series of simple geometric constructions based on the image circle diagram. The desired dielectric constant then follows from (1a) or (1b).

In the interest of generality we will proceed from the initial assumption that reference planes T and T' are located arbitrarily with respect to the sample. Eq. (12) is first rewritten in the form[4]

$$\Gamma' = \frac{a\bar{c}|\Gamma|^2 - b\bar{d}}{c\bar{c}|\Gamma|^2 - d\bar{d}} - \left[\frac{\bar{c}\bar{\Gamma} + \bar{d}}{c\Gamma + d}\right]\left[\frac{ad - bc}{c\bar{c}|\Gamma|^2 - d\bar{d}}\right]\Gamma, \quad (16)$$

where the bar over a quantity designates the complex conjugate and the primes have been omitted. One can determine the center of the image circle and its radius from (16) by inspection. Thus, when $|\Gamma|=1$, corresponding to a reactive termination in the output waveguide, the first term in (16) will be a complex constant and the magnitude of the second term will be a constant for all values of reactance. Referring to Fig. 2, the center of the image circle in the Γ' plane is

$$\Gamma_{C'} = \frac{a\bar{c} - b\bar{d}}{c\bar{c} - d\bar{d}} \quad (17)$$

and the radius is

$$R = \frac{4}{|c\bar{c} - d\bar{d}|}. \quad (18)$$

There are three points in the reflection plane which are of special interest. The iconocenter $\Gamma_{O'} = S_{11} = b/d$ is the map or image of $\Gamma = 0$ in the Γ' plane. There are a number of geometric constructions which can be used to obtain the iconocenter once the image circle, and its center are known.[3,6–8] All of these methods make use of

[4] H. F. Mathis, "Some properties of image circles," IRE Trans. on Microwave Theory and Techniques, vol. MTT-4, pp. 48–50; January, 1956.

[5] E. F. Bolinder, "Impedance and Power Transformations by the Isometric Circle Method and Non-Euclidean Hyperbolic Geometry," M.I.T. Res. Lab. of Electronics, Cambridge, Mass., Tech. Rept. 312; June 14, 1957.

[6] J. E. Storer, L. S. Sheingold, and S. Stein, "A simple graphical analysis of a two-port waveguide junction," Proc. IRE, vol. 41, pp. 1004–1013; August, 1953.

[7] F. L. Wentworth and D. R. Barthel, "A simplified calibration of two-port transmission line devices," IRE Trans. on Microwave Theory and Techniques, vol. MTT-4, pp. 173–175; July, 1956.

[8] G. A. Deschamps, "A variant in the measurement of two-port junctions," IRE Trans. on Microwave Theory and Techniques, vol. MTT-5, pp. 159–161; April, 1957.

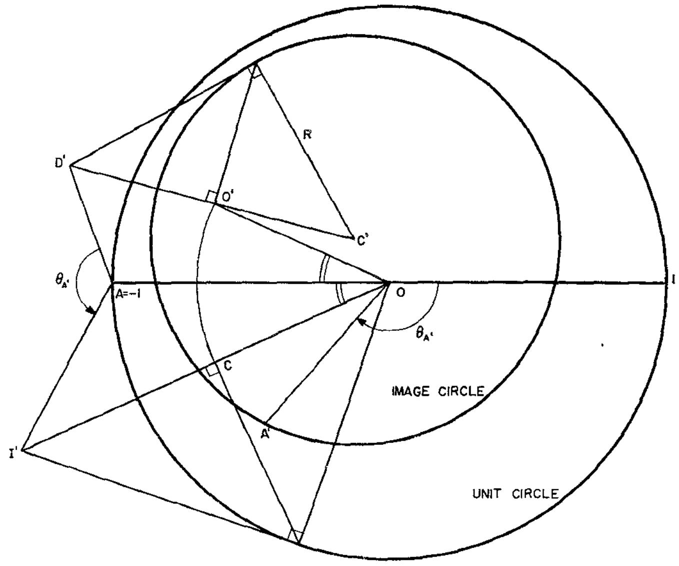

Fig. 2—Construction of $\Gamma_{D'}$ and $\Gamma_{I'}$, and determination of reference angle $\theta_{A'}$.

a calibrated short behind the network, and all but the last method referenced depend on a knowledge of the wavelength in the output waveguide. The other two points of interest in the reflection plane are $\Gamma_{D'}=a/c$ and $\Gamma_{I'}=-d/c$. These points have a special significance in the theory of the bilinear transformation and can be shown to mark the center of the isometric circles for the direct and inverse transformations, respectively.[9]

Graphical Analysis

Once $\Gamma_{O'}$ and $\Gamma_{C'}$ are determined, $\Gamma_{D'}$ can be readily constructed by inverting $\Gamma_{O'}$ with respect to the image circle as illustrated in Fig. 2. The proof of this statement is contained in

$$(\Gamma_{D'} - \Gamma_{C'})(\overline{\Gamma}_{O'} - \overline{\Gamma}_{C'}) = R^2, \tag{19}$$

which follows from (17), (18), and the condition, $ad-bc=4$. The construction of the point $\Gamma_{I'}$ follows in an analogous way, although in contrast to the previous case, the angle of $\Gamma_{I'}$ will depend on the choice of the output reference plane T. It is convenient at this point to locate T symmetrically with respect to T' so that $l_1=l_2$. Assuming for the moment that T has been so located, the symmetry of the network representing the dielectric then guarantees that $b=-c$, and $\Gamma_{I'}$ can be determined graphically by constructing the reciprocal of $\Gamma_{O'}$. This construction is also illustrated in Fig. 2. It is interesting to note that $\Gamma_{I'}$ is also the inverse with respect to the unit circle of $\Gamma_C=-\bar{c}/\bar{d}$, which is the map in the output Γ plane of the center of the image circle, $\Gamma_{C'}$, via the inverse transformation (9).

[9] E. F. Bolinder, "Impedance and polarization-ratio transformations by a graphical method using the isometric circles," IRE Trans. on Microwave Theory and Techniques, vol. MTT-4, pp. 176–180; July, 1956.

The location of T is accomplished by noting that when $b=-c$, the point $A=-1$ maps into A' at

$$\Gamma_{A'} = \frac{1+\Gamma_{D'}}{1+\Gamma_{I'}}\cdot \tag{20}$$

The reference point A' on the image circle is determined by the angle $\theta_{A'}=\arg\Gamma_{A'}$, which can be constructed from $\Gamma_{D'}$ and $\Gamma_{I'}$, as shown in Fig. 2. Thus, in order to guarantee symmetry one locates T at that position of the short in the output waveguide which establishes a voltage minimum at a distance $l=(\pi-\theta_{A'})\lambda_{0g}/4\pi$ from T' toward the generator in the input waveguide. It should be noted that this procedure establishes T only to within a multiple of half a wavelength. However, an approximate knowledge of the location of the sample suffices to remove any ambiguity.

At this point it is desirable to distinguish between the location invariant and the length invariant procedures. In the location invariant procedure, T' is located at an arbitrary point in the input waveguide, and T is determined from symmetry considerations as described above. The distance to the sample faces is calculated from $l_1=l_2=(w-d)/2$, where w and d are defined in Fig. 1 and are assumed to be known. It then remains to transform $\Gamma_{D'}$ and $\Gamma_{I'}$ to reference planes T_1 and T_2 at the sample faces. If $\Gamma_{D'}$ and $\Gamma_{I'}$ denote the isometric centers relative to planes T' and T, and Γ_D and Γ_I the corresponding quantities relative to planes T_1 and T_2, then it follows from (11) that

$$\Gamma_D = e^{2j\phi}\Gamma_{D'}, \qquad \Gamma_I = e^{-2j\phi}\Gamma_{I'}, \tag{21}$$

where $\phi=(w-d)\pi/\lambda_g$. The transformation of reference planes and the construction of Γ_E, which is the average of Γ and Γ_I, is apparent from Fig. 3. In the case of a TEM structure, the desired dielectric constant can be read directly from a Smith Chart overlay, in view of (1a) and (15), by constructing the point

$$\Gamma_F \equiv -1/\Gamma_E = \frac{\epsilon/\epsilon_0 - 1}{\epsilon/\epsilon_0 + 1}\cdot \tag{22}$$

This construction is also shown in Fig. 3.

In the length invariant procedure one makes the initial assumption that the front face of the sample can be located accurately by physical means, thus making $T'=T_1$. This amounts to a trivial distance determination, which, in rectangular waveguides, can be readily accomplished by mounting the sample in a shorting switch. Since the analysis proceeds from the assumption of symmetry, no distance measurement is required in the length invariant case and the actual location of reference plane T is of no interest. The points Γ_E and Γ_F are derived directly from $\Gamma_{D'}$ and $\Gamma_{I'}$ as determined in Fig. 2 without shifting reference planes.

When the loss tangent of the dielectric is relatively small, the graphical method will give rather poor per-

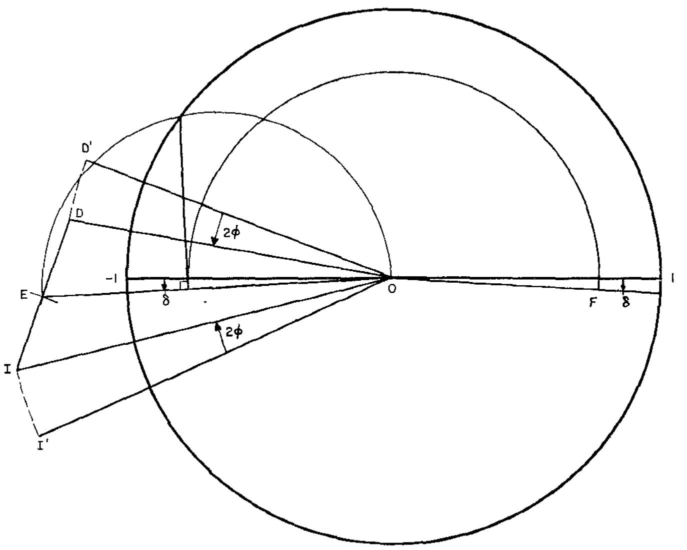

Fig. 3—Shift of reference planes and construction of Γ_F.

centage accuracy in the determination of ϵ''. In this case it is advisable to determine ϵ'' independently. This can be done most simply by using the already determined image circle to calculate the intrinsic insertion loss of the dielectric.[10] The necessary formulas are listed below for the convenience of the reader. If $\rho = |\Gamma_{C''}|$ is the distance of the center of the image circle from the origin of the reflection plane, it can be shown that

$$2\alpha d = \ln\left[\frac{\sqrt{(1+R)^2-\rho^2}+\sqrt{(1-R)^2-\rho^2}}{\sqrt{(1+R)^2-\rho^2}-\sqrt{(1-R)^2-\rho^2}}\right]. \quad (23)$$

Knowing ϵ', the desired ϵ'' is then obtained from either

$$\epsilon'' = 2\left(\frac{\alpha\lambda}{2\pi}\right)^2\sqrt{1+\left(\frac{2\pi}{\alpha\lambda}\right)^2\epsilon'} \text{ (TEM modes)}, \quad (24)$$

where λ is the free-space wavelength or

$$\epsilon'' \cong \frac{\alpha\lambda_{0g}}{\pi}\,\frac{\sqrt{[1+(\lambda_{0g}/\lambda_c)^2]\epsilon'-(\lambda_{0g}/\lambda_c)^2}}{1+(\lambda_{0g}/\lambda_c)^2} \text{ (H modes)}, \quad (25)$$

if

$$\epsilon'' \ll \epsilon' - \frac{1}{1+(\lambda_c/\lambda_{0g})^2}. \quad (26)$$

Appendix

It is well known that repeated bilinear transformations can be expressed in terms of a matrix product. Thus, if n linear networks are connected in cascade as shown in Fig. 4, the reflection matrix of the combination is given by

$$\begin{bmatrix} 2^{n-1}a & 2^{n-1}b \\ 2^{n-1}c & 2^{n-1}d \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\cdots\begin{bmatrix} a_n & b_n \\ c_n & d_n \end{bmatrix}. \quad (27)$$

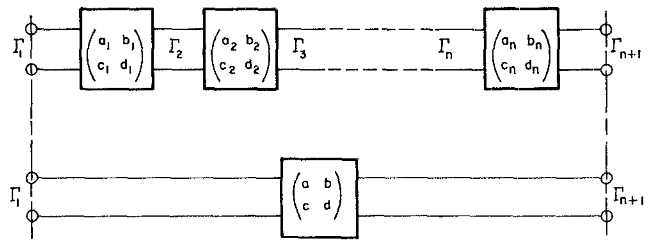

Fig. 4—Cascade connection of linear, bilateral two-port networks.

The factor of 2^{n-1} guarantees that $ad-bc=4$ if $a_id_i-b_ic_i=4$, $i=1, 2, 3, \cdots, n$. The problem of determining the scattering matrix of a cascade connection of networks is, therefore, reduced to a systematic and relatively direct procedure through the use of (27) and (10). In this application the reflection matrix bears a close resemblance to the transmission or T matrix,[11] as might have been anticipated from

$$T = \begin{bmatrix} a/2 & -c/2 \\ -b/2 & d/2 \end{bmatrix}. \quad (28)$$

[10] K. Tomiyasu, "Intrinsic insertion loss of a mismatched network," IRE Trans. on Microwave Theory and Techniques, vol. MTT-3, pp. 40–44; January, 1955.

[11] C. G. Montgomery, R. H. Dicke, and E. M. Purcell, "Principles of Microwave Circuits," M.I.T. Rad. Lab. Ser., McGraw-Hill Book Co., Inc., New York, N. Y., vol. 8, p. 150; 1948.